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COMMENTS ON THE IMPACT EFFECT†

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The relations of stereomechanical impact are proved by taking the limit as the impact time tends to zero. A generalized Kelvin formula for the change in kinetic energy during impact is given as an example. © 1999 Elsevier Science Ltd. All rights reserved.

Impact theory is based on the assumption that an impact force has an effect only over a very short time interval, though the magnitude of that force is considerable. Consequently, the position of the system hardly changes during impact, while the velocities have finite increments. In the limit (as the impact time tends to zero) we obtain a stereomechanical model of impact. However, this limit has not been proved with any rigour, either in the classical textbooks (see [1, 2], for example) or in contemporary works ([3], for example). The purpose of this note is to fill this gap.

Consider the motion of a mechanical system with n degrees of freedom. Let $q_1, \ldots, q_n$ be generalized coordinates, T the kinetic energy and $Q_1, \ldots, O_n$ the generalized forces. Suppose further forces $F_l(t), \ldots, F_n(t)$ operate on the system, defined as follows: $F_k(t) = 0$ for $t \leq 0$ and $t > \varepsilon$ (ε is a positive parameter, which will be made to approach zero), $F_k(t) = I_k/\varepsilon$ for $0 < t \leq \varepsilon$, where I_k are certain constants. In the limit, as $\varepsilon \to \infty$, obviously, $F_k(t) = I_k\delta(t)$, where δ is the Dirac delta function.

Under the given assumptions, the motion of the mechanical system is determined from Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k + F_k, \quad 1 \leq k \leq n \tag{1}$$

The kinetic energy T and the generalized forces Q are assumed to be smooth functions of $\dot{q}$, q and t the function T being such that the Legendre transformation is defined in terms of the velocities $\dot{q}_1, \ldots, \dot{q}_n$. Let $p_k = \partial T/\partial \dot{q}_k$ be the momenta associated with the coordinates q_k.

We consider the motion $q_\varepsilon(t)$ with initial data independent of ε

$$q_k(0) = q_k^0, \quad \dot{q}_k(0) = v_k^-$$

Let p_k^- be the value of the canonical moment at $t = 0$. Then

$$\lim_{\varepsilon \to 0} q_\varepsilon(t) = \hat{q}(t)$$

exists, the limiting function satisfying Eqs (1) with $F = 0$ for $t < 0$ and $t > 0$, and

$$\lim_{t \to -0} \hat{q}_k(t) = q_k^0, \quad \lim_{t \to -0} \dot{\hat{q}}_k(t) = v_k^+ \tag{2}$$

The momenta p_k^+ corresponding to the values $t = 0$, $q = q_0$, $\dot{q} = \upsilon^+$ are related to p_k^- by the simple equation

$$p_k^+ - p_k^- = I_k \tag{3}$$

which is the basis of the theory of stereomechanical impact.

We will now prove this assertion. Changing from Lagrange's equations (1) to Hamilton's equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} + Q_k + F_k, \quad 1 \leq k \leq n \tag{4}$$

where H is the kinetic energy represented in canonical variables, we replace the time $t \to \tau$ in (4) by $F_k(\tau)$ will be non-zero over the interval $0 < \tau \leq l$. Denoting differentiation with respect to τ by a prime, we rewrite Hamilton's equations

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$$q_k' = \varepsilon\frac{\partial\tilde{H}}{\partial p_k}, \quad p_k' = -\varepsilon\frac{\partial\tilde{H}}{\partial q_k} + \varepsilon\tilde{Q}_k + F_k(\tau) \tag{5}$$

Here $\tilde{H}$ and $\tilde{Q}_k$ are the functions H and Q_k in which the time t, which appears explicitly, is replaced by $\varepsilon\tau$. Since the right-hand sides of Eqs (5) depend continuously on the small parameter ε, by Poincaré's theorem their solutions (with fixed initial data q^0, p^- at $t = 0$) can be written in the following form for values $0 < \tau \leqslant 1$

$$q_k(\tau) = q_k^0 + \varepsilon(\cdot), \quad p_k(\tau) = p_k^- + I_k\tau + \varepsilon(\cdot) \tag{6}$$

The expressions in brackets are continuous functions of τ and ε. Putting $\tau = 1$ (the finite time of impact) and letting ε tend to zero, we obtain relations (2) and (3).

We assume that the kinetic energy does not depend explicitly on time and has the obvious form: $T = T_2 + T_1 + T_0$, where T_s is the homogeneous form of degree s. The change of energy during impact is given by Kelvin's formula

$$\Delta H = (I, (\upsilon^+ + \upsilon^-)/2) \tag{7}$$

where $H = T_2 - T_0$ (the Hamilton function), I is the impact momentum defined by formula (3) and $\upsilon^-(\upsilon^+)$ is the velocity of the system before (after) impact.

Formula (7) is usually proved for systems of free points and for solids (see [2, 3]). However, Kelvin's formula is valid in the most general case where the relations imposed on the system area time-dependent ($T_1 \neq 0$). It is a curious identity for quadratic forms.

We will show how to derive the generalized Kelvin formula by the above method. We use the generalized theorem of the change in kinetic energy

$$dH = \Sigma Q_k dq_k + \Sigma F_k dq_k$$

Using the new time $\tau = t/\varepsilon$, we obtain the increment of Hamilton's function during the time of action of the impact pulse

$$\Delta H = \int_0^1 (Q, q')d\tau + \int_0^1 (I, \dot{q})d\tau \tag{8}$$

By virtue of Eqs (5), the derivative q' is small together with ε. Thus, as $\varepsilon \to 0$ the first term on the right-hand side of Eq. (8) tends to zero. According to Eqs (6), when $\varepsilon = 0$ the velocity $\dot{q}$ is a linear function of τ. Since the integral of a linear function is equal to half the sum of its values at the ends of the interval of integration, the generalized Kelvin formula follows immediately from (8).

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